

# Corrigendum for Strategic Interaction and Interstate Crises: A Bayesian Quantal Response Estimator for Incomplete Information Games \*

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The uniqueness claim of Lemma 1 from Esarey, Mukherjee and Moore (2008) has been disproved. We were made aware of this refutation when we received an anonymous reviewer report for a new manuscript (not written by us) that relied on our Lemma 1. The anonymous reviewer report, provided to us by the editors of *Political Analysis*, proposed a case that contradicted Lemma 1 by showing multiple BQRE solutions for a single finite extensive form game. We were able to verify the validity of that counterexample. Therefore, we conclude that the uniqueness claim of Lemma 1 is false and consequently that Bayesian Quantal Response Equilibria (BQRE) are not guaranteed to be unique under the conditions that we specified. The explanation is similar to one proffered by Jo (2011) for multiplicity in Perfect Bayesian Equilibrium models proposed in Lewis and Schultz (2003). In this corrigendum, we present the counterexample and briefly discuss the implications for estimating BQRE models on data.

## Counterexample to Lemma 1

We begin by quoting the lemma in question from Esarey, Mukherjee and Moore (2008, p. 261):

**Lemma 1.** *There exists a unique BQRE solution for each choice probability for finite extensive form games of incomplete information when the players (i) have prior beliefs about their opponent's private information that are fixed and deterministic, (ii) update their posterior beliefs by using Bayes's rule and (iii) have affine payoffs at every terminal node of the game.*

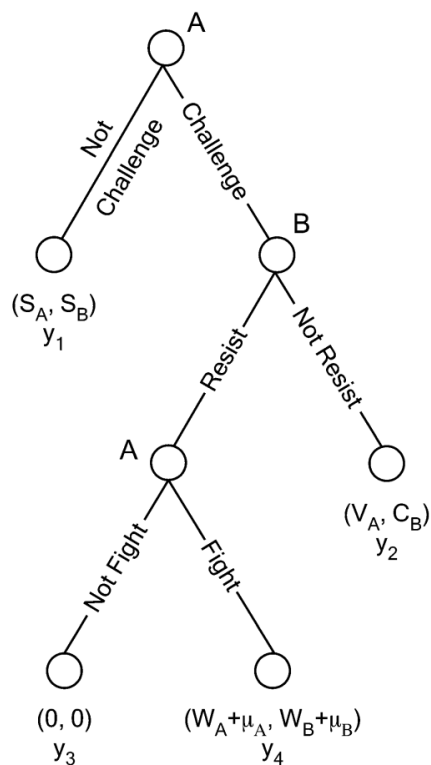
This lemma can be disproved by a single instance of a finite extensive form game of

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incomplete information with multiple BQRE solutions under the stated conditions. The analysis in Esarey, Mukherjee and Moore (2008) focuses on the crisis bargaining game in Figure 1, and although Lemma 1 is not limited to this situation a counterexample from this game suffices to challenge the lemma. This bargaining game has a long history in the literature pertaining to strategic choice; for example, it appears in Lewis and Schultz (2003).

Figure 1: Crisis Bargaining Game



The anonymous reviewer proposed the following set of payoffs for the crisis bargaining game:

$$(S_A, S_B) = (10, 0)$$

$$(V_A, C_B) = (8, 5)$$

$$(W_A, W_B) = (-5, 7)$$

The types (values of  $\mu_A$  and  $\mu_B$ ) for the players are given by:

$$\mu_A = \begin{cases} 10 & \text{with probability 0.25} \\ -10 & \text{with probability 0.75} \end{cases}$$

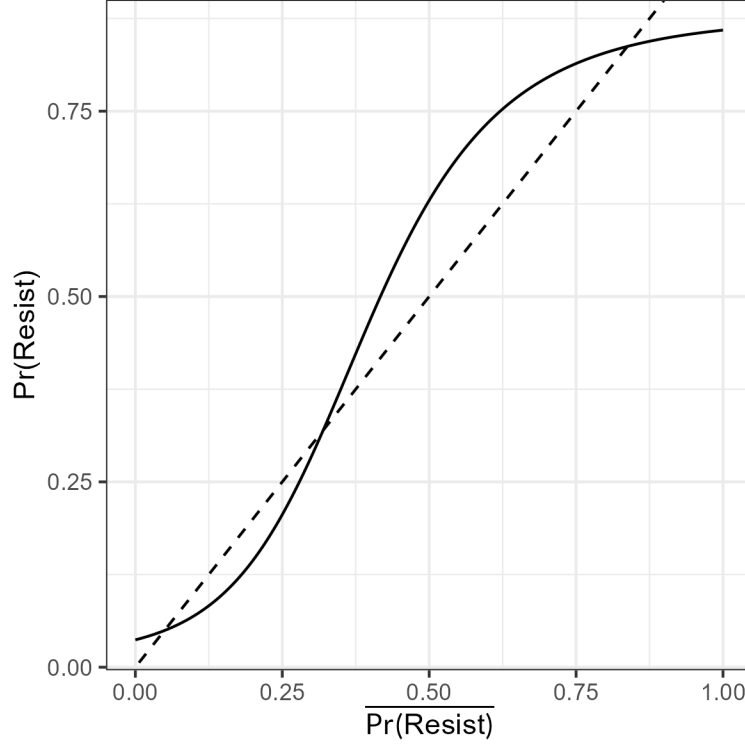
$$\mu_B = 0$$

In this situation, Player B has the opportunity to update their beliefs about Player A's type upon observing a challenge. Player A's decision to fight (or not) is a simple function of their payoffs, a function that produces a probability  $\Pr(F)$  according to the usual quantal response logistic formula. Player A's decision to challenge (or not) is also a simple function of their payoffs (producing a probability  $\Pr(C)$ , but weighted by their expectation of Player B's probability of choosing to resist. This probability of resistance  $\Pr(R)$  is a function of Player B's payoffs, his or her beliefs about  $\mu_A$ , and his or her expectations about  $\Pr(F)$ . Thus, we determine equilibria by recursively solving for  $\Pr(C)$  and  $\Pr(F)$  given a candidate  $\overline{\Pr(R)}$ , then solving for  $\Pr(R)$  given the solutions for  $\Pr(C)$  and  $\Pr(F)$ ; this becomes the new  $\overline{\Pr(R)}$ . The process is repeated until  $\Pr(R) - \overline{\Pr(R)} \approx 0$  to an appropriate numerical tolerance.

We show the relationship between  $\overline{\Pr(R)}$  and  $\Pr(R)$  for the proposed payoffs and types in Figure 2. Candidate equilibria are points where the solid recurrence line intersects the diagonal  $y = x$  line. Although there are three intersections, one is unstable: when  $\Pr(\text{Resist}) \approx 0.32$ , any minor perturbation will tend to push resistance to higher or lower levels until one of the other equilibria is reached. The other two cases,  $\Pr(\text{Resist}) \approx 0.0495$  and  $\Pr(\text{Resist}) \approx 0.837$ , are stable. The full characterization of these equilibria is:

*Equilibrium 1:*  $\Pr(F|\mu_A = 10) = 0.993$ ,  $\Pr(F|\mu_A = -10) = 3.06 \times 10^{-7}$ ,  $\Pr(C|\mu_A = 10) = 5.01 \times 10^{-7}$ ,  $\Pr(C|\mu_A = -10) = 3.92 \times 10^{-7}$ ,  $\Pr(R) = 0.0495$ ,  $\Pr(\mu_A = 10|C) = 0.561$

Figure 2: Recurrence Plot for  $\Pr(\text{Resist})$



*Equilibrium 2:*  $\Pr(F|\mu_A = 10) = 0.993$ ,  $\Pr(F|\mu_A = -10) = 3.06 \times 10^{-7}$ ,  $\Pr(C|\mu_A = 10) = 0.00129$ ,  $\Pr(C|\mu_A = -10) = 2.01 \times 10^{-5}$ ,  $\Pr(R) = 0.837$ ,  $\Pr(\mu_A = 10|C) = 0.985$

Qualitatively, the first equilibrium is one where Player B rarely resists. This provides little incentive for either type of Player A to challenge, meaning that they do so very rarely and with basically equal probability. The second equilibrium is one where Player B often resists. This provides a greater incentive for the  $\mu_A = 10$  type of Player A to challenge—albeit still rarely—leading to beliefs consistent with that strategy.

The existence of two stable BQRE solutions to this problem suffices to contradict the uniqueness claim of Lemma 1.

## Implications

When using a BQRE model on data, it is possible that a single set of payoffs (and thus a single set of parameters mapping observations into utility space) is consistent with more

than one pattern of behavior. This presents difficulties when estimating parameters mapping observations into payoffs. The likelihood portion of the posterior distribution of parameters for an observation indexed by  $p$  is:

$$L = \prod_p \left[ (1 - \Pr(C))^{y_1^p} \right] \times \left[ (\Pr(C) \times (1 - \Pr(R)))^{y_2^p} \right] \times \\ \left[ (\Pr(C) \times \Pr(R) \times (1 - \Pr(F)))^{y_3^p} \right] \times \\ \left[ (\Pr(C) \times \Pr(R) \times \Pr(F))^{y_4^p} \right]$$

If there are multiple viable values of the component probabilities ( $\Pr(C)$ ,  $\Pr(R)$ , and  $\Pr(F)$ ) consistent with the same payoffs, this creates an ambiguity in how the likelihood should be calculated for a given set of responses  $y = \{y^1, y^2, \dots, y^N\}$ . An analyst using our model should be mindful of this possibility.

We speculate that the most straightforward reaction to the possibility of multiple equilibria is to implement a solver that enables detection of multiple equilibria consistent with a pattern of payoffs; the `uniroot` function in base R (R Core Team, 2026) accomplishes this (and was used in creating this corrigendum). Our original analysis (both for generating Monte Carlo data and estimating models on this data) used a simple iterative root-finding approach that would ignore the possibility of multiple equilibria and report only one. If multiple equilibria are detected in the estimation process by `uniroot` or some other method, we see two ways to proceed.

1. A more conservative approach would be to simply stop the estimation process and return an error whenever a parameter vector consistent with multiple equilibria is detected. For many problems, the BQRE solution will be unique and this error will not be encountered. If it is encountered, the modeling approach should be discarded.
2. A less conservative approach would alter the likelihood function to use the equilibrium consistent with the highest likelihood for any observation where multi-

ple equilibria are detected. This approach would essentially select the equilibrium most consistent with the data whenever ambiguity is present. We speculate that this would make the likelihood space more complex and less amenable to gradient-based optimization, necessitating a fully Bayesian sampling approach using a technique robust to multimodal posterior distributions. Fully exploring the potential consequences of this choice is beyond the scope of this corrigendum, and may present difficulties that we cannot foresee at this time.

We thank the editors of *Political Analysis* and the anonymous reviewer for bringing this issue to our attention.

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